A NEW TEST OF COMPOUND SYMMETRY

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1. Summary. If x_1 and x_2 have a bivariate normal distribution with a correlation coefficient ρ and the same standard deviations σ for both, then it is well known and easy to check that $x_1 + x_2$ and $x_1 - x_2$ are uncorrelated, which forms the basis of Pitman's well-known test of H_0: $\sigma_1 = \sigma_2$, for a bivariate normal population, in terms of the correlation coefficient r between $x_1 + x_2$ and $x_1 - x_2$ in a random sample of size, say, n, from this population. Starting from this test which has a number of reasonably good properties, and then using the union-intersection principle [1, 2], a test is obtained for compound symmetry, i.e., for H_0: $\sigma_{11} = \sigma_{22} = \dots = \sigma_{pp}$ and all σ_{ij}'s are equal ($i \neq j = 1, 2, \dots, p$), where σ_{ij} is any element of the covariance matrix Σ of a p-variate normal population.

2. Test Construction. Let $\rho(x_1, x_2)$ denote the population correlation coefficient between x_1 and x_2 and $r(x_1, x_2)$ the same in a random sample of size, say n, from that population. Then, for a set of stochastic variables $\underline{x}'(1 \times p)(= x_1, \dots, x_p)$ having a p-variate normal distribution, notice that for any arbitrary non-null vector $\underline{a}'(1 \times p)$, $\underline{a}'(1 \times p)\ \underline{x}(p \times 1)$ and $\sum_{i=1}^{p} x_i$ have a bivariate normal distribution and, now letting $\sum_{i=1}^{p} a_i = 0$,

consider the hypothesis: $\rho(\sum_{i=1}^{p} x_i, \underline{a}'\underline{x}) = 0 = H_{0\underline{a}}$ (say). Next notice that

(2.1) H_0: all σ_{ii}'s are equal and all σ_{ij}'s are equal $(i \neq j = 1, \ldots, p)$

$$= \bigcap_{\underline{a}} H_{0\underline{a}} = \bigcap_{\underline{a}} [\, \rho\, (\sum_{i=1}^{p} x_i, \underline{a}'\underline{x}) = 0 \,] ,$$

where $\underline{a}'(1 \times p)$ is any arbitrary row vector subject to $\sum_{i=1}^{p} a_i = 0$.

Now going back to $H_{0\underline{a}}$, we have for this hypothesis the Pitman critical region, say $W_{\underline{a}}(\alpha)$, of size α, given by

(2.2) $$W_{\underline{a}}(\alpha): \quad r^2(\sum_{i=1}^{p} x_i, \underline{a}'\, \underline{x}) \geq r_{\alpha}^2\, (n-2),$$

where $r_{\alpha}(n-2)$ is the upper $\alpha/2$-point of the central r-distribution in random samples of size n.

Hence, by the union-intersection heuristic principle [1, 2] we have, for $H_0 (= \bigcap_{\underline{a}} H_{0\underline{a}})$, the critical region $W(\beta)$ of size β given by

(2.3) $$W(\beta): \bigcap_{\underline{a}} [\, r^2(\sum_{i=1}^{p} x_i, \underline{a}'\underline{x}) \geq r_{\alpha}^2(n-2) \,] ,$$

$$\text{i.e., } \operatorname{Sup}_{\underline{a}}\, r^2(\sum_{i=1}^{p} x_i, \underline{a}'\underline{x}) \geq r_{\alpha}^2(n-2)$$

$$\text{i.e., } \operatorname{Sup}_{\underline{a}}\, r^2(\sum_{i=1}^{p} x_i, \sum_{i=1}^{p-1} a_i(x_i - x_p)\,) \geq r_{\alpha}^2(n-2) ,$$

remembering that $\sum_{i=1}^{p} a_i = 0$, i.e., $a_p = -\sum_{i=1}^{p-1} a_i$.

It is easy to check that

$$(2.4) \qquad \mathrm{Sup}_{\underline{a}}\ r^2\left(\sum_{i=1}^{p} x_i,\ \sum_{i=1}^{p-1} a_i\,(x_i - x_p)\right)$$

= square of the sample multiple correlation between $\sum_{i=1}^{p} x_i$ and the $(p - 1)$-set of variables $(x_1 - x_p)$, $(x_2 - x_p)$, ..., $(x_{p-1} - x_p)$ =

$$R^2\left[\sum_{i=1}^{p} x_i \text{ and } (x_1 - x_p), (x_2 - x_p), \ldots, (x_{p-1} - x_p)\right] = R^2 \text{ (say)}.$$

Notice that, since the new p-set also has the p-variate normal distribution, this R has the well-known multiple correlation distribution with degrees of freedom $p - 1$ and $n - p$ and a non-centrality parameter which is the population multiple correlation between $\sum_{i=1}^{p} x_i$ and the $(p - 1)$-set above and which let us call ρ^2. It is easy to check that $\rho = 0$, i.e., that R has the central multiple correlation distribution, if and only if

$\rho(\sum_{i=1}^{p} x_i,\ x_i - x_p) = 0$ $(i = 1, 2, \ldots, p - 1)$, i.e., if and only if

$\rho\,(\sum_{i=1}^{p} x_i,\ \underline{a}'\underline{x}) = 0$ (for all non-null $\underline{a}$ subject to $\sum_{i=1}^{p} a_i = 0$).

We have thus, for testing compound symmetry, the critical region $W(\beta)$ of size β given by

$$(2.5) \qquad R\left[\sum_{i=1}^{p} x_i \text{ and } (x_1 - x_p), \ldots, (x_{p-1} - x_p)\right] \geq R_\beta\,(p - 1, n - p),$$

when R_β is the upper β-point of the well-known central multiple correlation distribution.

It can be checked after some little algebra that in terms, respectively, of the elements of the sample and population covariance matrices S and Σ, R and ρ will be given by

$$R^2 = 1 - p \sum_{i=1}^{p} z_i z^i / \left(\sum_{i=1}^{p} z_i\right) \left(\sum_{i=1}^{p} z^i\right), \tag{2.6}$$

$$\rho^2 = 1 - p \sum_{i=1}^{p} \zeta_i \zeta^i / \left(\sum_{i=1}^{p} \zeta_i\right) \left(\sum_{i=1}^{p} \zeta^i\right),$$

when

$$z_i = \sum_{j=1}^{p} s_{ij}, \quad z^i = \sum_{j=1}^{p} s^{ij},$$

$$\zeta_i = \sum_{j=1}^{p} \sigma_{ij} \text{ and } \zeta^i = \sum_{j=1}^{p} \sigma^{ij} .$$

Note that $s_{ij} = s_{ji}$, $s^{ij} = s^{ji}$, $\sigma_{ij} = \sigma_{ji}$ and $\sigma^{ij} = \sigma^{ji}$.

The power properties of this test will be discussed in a later note.

References

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2. S. N. Roy and R. C. Bose, "Simultaneous confidence interval estimation," Annals of Mathematical Statistics, Vol. 24 (1953), pp. 513-36.